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LETTER TO THE EDITOR

Space-periodically driven codimension-two bifurcations

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Abstract. An amplitude equation is derived for a one-dimensional extended dynamical system undergoing a codimension-two bifurcation subjected to a weak near-resonant spatially periodic forcing. It is proved that in some appropriate limit the dynamics is of the propagative phase type and it is governed by the time-dependent sine-Gordon equation with weak dissipation.

During the last few years a problem of high current interest has been that of symmetry-breaking instabilities leading to pattern formation in non-equilibrium systems. Such instabilities in non-linear systems driven far from thermodynamic equilibrium are the analogue of equilibrium phase transitions in condensed matter physics with the difference that in these systems the invariance under time translations may also be spontaneously broken, leading to states varying periodically in time.

In many non-equilibrium systems an instability often consists of a transition from an unstructured uniform state to one varying periodically in time or space; in other words, the first instability that occurs can be either stationary or oscillatory. The appearance of periodic structures in these systems driven externally by a set of time-independent control parameters corresponds to a bifurcation, characterised by the undamping of one or several normal modes as the control parameters are varied, and the breaking of a symmetry. When the number of these critical modes is finite, their amplitudes are governed in the vicinity of the bifurcation by non-linear partial differential equations (amplitude equations) in which the growth rates of the linear theory have been renormalised by non-linear terms and describe slow modulations in time and space of the periodic structure envelope.

Once the instability has given rise to a spatially periodic pattern in a system with translational invariance, one is naturally led to ask what is the mechanism responsible for the selection of a particular pattern wavevector from the band of possible wavevectors. A partial answer to this question can be given if the translational symmetry of the system is broken externally by a spatially periodic forcing and by examining the response of the system to forcing at a wavevector that may be different from the naturally selected one. This has been the approach followed by Lowe *et al* (1983) and Lowe and Gollub (1985) whose experimental results in a convecting nematic fluid subjected to a spatially periodic potential difference led to the discovery of soliton lattices in connection with the selection of patterns resulting from an electrohydrodynamic instability and yielded a surprisingly rich variety of modulated structures,

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including one-dimensional commensurate phase-locked patterns, incommensurate states characterised by soliton-like phase disturbances and two-dimensional structures like crystal lattices.

From the theoretical point of view the effect of a spatially periodic forcing has already been studied by Kelley and Pal (1978) in the context of thermal convection at the onset of a stationary instability, but only commensurate states were considered, and recently by Coullet (1986) who derived an amplitude equation for the slowly varying envelope of a one-dimensional periodic pattern under the presence of a weak periodic forcing. He showed that the competition between periodicities can lead to a commensurate-incommensurate (CI) transition mediated by time-independent phase solitons arising from a 'diffusive' sine-Gordon equation in agreement with the experiments of Lowe and co-workers. As has already been pointed out by Coullet (1986), if one considers a propagative phase dynamics instead of a diffusive one the UI transition can be mediated by dynamical sine-Gordon solitons whose eventual non-linear evolution may lead to soliton turbulence.

The aim of this letter is to consider one-dimensional extended dynamical systems subjected to a weak spatially periodic forcing, leading naturally to a propagative phase dynamics which in some appropriate limit reduces to the time-dependent sine-Gordon equation.

The natural framework for studying the above situation is provided by the study of an amplitude equation derived in the neighbourhood of a point in parameter space where a codimension-two bifurcation occurs arising from the intersection of a codimension-one line associated with a stationary instability (simple zero eigenvalue) and a codimension-one line associated with an oscillatory instability (Hopf bifurcation).

Following the above reasoning, let us consider a one-dimensional extended dynamical system undergoing a codimension-two bifurcation at a wavenumber $k = k_c$. Let $U(x, t)$ be a set of real scalar fields describing the physical system and modelling its pattern-forming transitions. We assume that the U obey an evolution equation of the form

$$MU_t = LU + N(U) \tag{1}$$

where L and N represent linear and non-linear differential operators, respectively, depending on a set of parameters μ_i, $i = 1, 2, \ldots$, and M is an invertible linear operator. For instance, the equations of bi-dimensional convection with free-free boundaries are easily cast into the form (1) where U represents the stream function and the temperature and parameters are the Rayleigh and Prandtl numbers.

In the absence of a spatially periodic forcing the system considered is assumed to be invariant under spacetime translations and space reflections. The critical situation considered occurs when two parameters, say μ_1 and μ_2, take critical values μ_{1_c}, μ_{2_c} such that the operator L in Fourier space has a doubly degenerate zero eigenvalue at $|k| = k_c$ with an associated Jordan matrix

$$\mathbb{J} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2}$$

whilst the rest of its eigenvalues remain with a strictly negative real part for all k. By using the fact that the spatial symmetries of the problem can be identified with the $SO(2) \times Z_2 \simeq O(2)$ symmetry, the critical eigenvalues become doubled and there the critical subspace becomes four dimensional. Let us denote by $\psi_{1\pm} = \phi_1 \exp(\pm ik_c x)$

and $\psi_{2\pm} = \phi_2 \exp(\pm ik_c x)$ the critical modes. Then the restriction of L to the subspace spanned by $\{\psi_{1\pm}, \psi_{2\pm}\}$ is the double Jordan matrix $\mathbb{J}\otimes\mathbb{J}$ such that $L\psi_{1\pm} = 0$, $L\psi_{2_\pm} = \psi_{1\pm}$.

In the asymptotic regime of long times the non-linear dynamics reduces to the centre manifold which is locally tangent to the critical subspace. Therefore we look for U in the form

$$U = A(x, t)\psi_{1\pm} + B(x, t)\psi_{2+} + \text{cc} + U \tag{3}$$

where U represents the centre-manifold contribution consisting of linear and non-linear corrections which are assumed to be small for $|\mu_1 - \mu_{1_c}| \ll 1$, $|\mu_2 - \mu_{2_c}| \ll 1$, and A and B represent two slowly varying complex fields necessary to describe the onset of the instability. The fields A and B obey an amplitude equation which can be derived by standard methods (Coullet and Spiegel 1983):

$$\begin{pmatrix} A \\ B \end{pmatrix}_t = \mathscr{L}\begin{pmatrix} A \\ B \end{pmatrix} + \begin{pmatrix} N_1(A, B, \bar{A}, \bar{B}) \\ N_2(A, B, \bar{A}, \bar{B}) \end{pmatrix}. \tag{4}$$

The differential linear operator $\mathscr{L}$ is nothing other than the operator version of the Arnold–Jordon matrix corresponding to the critical matrix $\mathbb{J}$. Using the symmetry $x \to -x$ $\mathscr{L}$ becomes

$$\mathscr{L} = \begin{pmatrix} 0 & 1 \\ \mu + f(\partial_{xx}) & \nu + g(\partial_{xx}) \end{pmatrix} \tag{5}$$

where μ and ν are small parameters which measure the deviation from threshold. The non-linear terms in (4) are determined by using the fact (Elphick *et al* 1986) that N_1 and N_2 are equivariant under the Lie group generated by $\mathbb{J}\otimes\mathbb{J}$ modulo non-resonant terms (terms that can be eliminated by a smooth non-linear change of variables). Using the last fact with a judicious choice of non-resonant terms, together with the SO(2) symmetry, we readily obtain

$$N_1 = 0$$
$$N_2 = AQ(|A|^2) + BP_1(|A|^2, \bar{A}B - A\bar{B}) + \bar{B}A^2P_2(|A|^2, \bar{A}B - A\bar{B}) \tag{6}$$

where Q, P_1 and P_2 are polynomials in their arguments.

When one takes into account the effect of a weak spatially periodic forcing with a wavenumber k_e, the translational invariance becomes the discrete invariance $x \to x + N2\pi/k_e$. In the case of resonant forcing $k_e = (n/m)k_c$ the discrete translational invariance implies the invariance of the amplitude equations under the transformations $A \to A\exp(i2\pi/n)$, $B \to B\exp(i2\pi/n)$ which in turn implies that N_2 contains a SO(2) symmetry-breaking term which at leading order becomes

$$c_1\bar{A}^{n-1} + c_2\bar{B}\bar{A}^{n-2} \tag{7}$$

where c_1, and c_2 are constants which generally scale as the mth power of the forcing amplitude. When one considers the misfit q between the external and natural periodicities ($k_e = (n/m)(k_c + q)$, $q \ll k_c$) (7) is modified by the replacements $c_1 \to c_1\exp(inqx)$, $c_2 \to c_2\exp(inqx)$.

Using the above results, the equations for the fields A and B at the leading order ($n \leqslant 4$) are

$$A_t = B$$
$$B_t = \mu A + \nu B + A_{xx} + B_{xx} + a|A|^2A + b\bar{B}A^2 + cB|A|^2$$
$$+ dB(\bar{A}B - A\bar{B}) + (c_1\bar{A}^{n-1} + c_2\bar{B}\bar{A}^{n-2})\exp(inqx) \tag{8}$$

which represent universal amplitude equations describing the onset of a codimension-two bifurcation in an extended one-dimensional system when it is subjected to a weak near-resonant spatially periodic forcing. Equation (8) can be further simplified by considering its asymptotic form (Arneodo *et al* 1985) which amounts to scale $A \to \varepsilon^{\alpha_1} A$, $B \to \varepsilon^{\alpha_2} B$, $t \to \varepsilon^{\alpha_3} t$, $x \to \varepsilon^{\alpha_4} x$, $\mu \to \varepsilon^{\alpha_5}\mu$, $\nu \to \varepsilon^{\alpha_6}\nu$ and retain the leading-order contributions in powers of ε. One readily obtains, after the transformation $A \to A\exp(\mathrm{i}qx)$, the following asymptotic equation for A:

$$A_{tt} - \nu A_t = (\mu - q^2)A + 2\mathrm{i}qA_x + A_{xx} + a|A|^2A + c_1\bar{A}^{n-1} \tag{9}$$

which is nothing other than the propagative-dissipative version of the equation derived by Coullet (1986) to describe the effect of a weak forcing at the onset of a codimension-one stationary bifurcation. In the limit of large dissipation we recover Coullet's equation. It is worth remarking that, although the translational invariance has been broken externally, equation (9) still possesses this invariance since A is assumed to vary on a scale much larger than $\max(2\pi/k_c, 2\pi/k_e)$.

To conclude, let us take $\mu > 0$, $\nu < 0$ and set for convenience $a = -1$. Then in the limit μ large, ν small (more precisely $\mu \gg q^2$, $c_1^{2/4-n}$, $|\nu| \ll \sqrt{\mu}$) we expect that the dynamics of A will be mainly dominated by its phase. Making the ansatz $A = R\exp(\mathrm{i}\theta)$ we extract from (9) the following phase equation:

$$\theta_{tt} - \nu\theta_t = \theta_{xx} - c_1\mu^{(n-2)/2}\sin(n\theta). \tag{10}$$

Therefore we have shown that the dynamics in the vicinity of a codimension-two bifurcation under the presence of a weak near-resonant forcing is governed by a propagative-dissipative amplitude equation with O(2) symmetry-breaking terms (equation (8)) which, when put into its asymptotic form, leads in some appropriate limit to a propagative phase dynamics described by the time-dependent sine-Gordon equation with a small damping term, measuring weak dissipation.

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